

# Counting bats

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Assume  $G$  is an infinite graph (“the cave”) which is recurrent for the simple random walk (SRW). Several independent walkers (“the bats”) are performing SRW on  $G$  simultaneously with the same clock with starting vertex  $o$ .  $G$  is not known to you (hence the cave metaphor), it is too dark to see  $G$ ). The only information given to you is the set of return times to  $o$ , though you do not know how many walkers returned at any given time, only if this number is 0 or positive. Can you almost surely tell how many walkers are there, by only observing the times  $o$  is occupied?

**Theorem.** *Almost surely it is possible to tell how many walkers are there, by observing the times  $o$  is occupied.*

Formally, there is a function  $\mathcal{A} : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{N}$  which, given the visits of the walkers outputs their number, and is correct with probability 1. Again,  $\mathcal{A}$  does not depend on the graph. For more bat-related results, see our paper [1].

*Corollary.* There is no pair of recurrent infinite graphs so that the return times of two independent SRW’s on one of the graphs are absolutely continuous to the return times of one SRW on the other graph.

*Problem.* The algorithm in the proof seems far from efficient. Give lower and upper bounds and suggest improved or optimal algorithms.

(“the algorithm” here is simply the function  $\mathcal{A}$ , which is not really an algorithm in the computer sense: it does not “run” or “stop”. Nevertheless one can find reasonable algorithmic versions of the problem and investigate them)

*Problem.* We do not know if reversibility is important (it is definitely used in our proof). So we ask: is there an algorithm that gives the right number of walkers for any recurrent Markov chain, without knowledge of the Markov chain?

## Proof

The first step is to reconstruct the distribution of returns of a *single* walker, no matter how many walkers one actually examines.

**Lemma 1.** *There is an algorithm to reconstruct*

$$p(n) = \mathbb{P}(\text{a single walker returns to } o \text{ at time } n)$$

*with no knowledge of the graph structure.*

*Proof.* We fix some large  $T_1$  and wait till you see a time interval of length  $\geq T_1$  with no visits to  $o$  and look at the first return after this long returns-free interval, denoted by  $s_1$ . Let  $E_1$  be the event that there is a return at  $s_1 + n$ . Continue similarly: choose some large  $T_2$ , let  $s_2$  be the first time after  $s_1 + n$  when a returns-free intervals longer than  $T_2$  finished, and let  $E_2$  be the event that there is a return at  $s_2 + n$ . Etc. For concreteness, fix  $T_i = 2^i$ .

Write now  $E_i = G_i \cup B_i$  where  $G_i$  (the “good” event) is the event that the walker which returned at time  $s_i$  also returned at  $s_i + n$ , and  $B_i$  (the “bad” event) is the event that another walker returned at time  $s_i + n$ . We would have liked to sample  $G_i$ , which are exactly i.i.d. variables with probability  $p(n)$ , but we can only sample  $E_i$ . So we need to show that  $B_i$  are rare.

The key observation follows from a quantitative non concentration of return times established in [3]: on any graph, the probability that a walker returned for the first time at time  $t$ , conditioned on not having returned before  $t$ , is  $\leq (C \log t)/t$ . Denote therefore  $B_i(j)$  the event that it is the  $j^{\text{th}}$  walker that returned at time  $s_i + n$  ( $B$  is a “bad” event, so we assume that the  $j^{\text{th}}$  walker did not return at time  $s_n$  or that there were two returns at  $s_n$ ). Let  $r$  be the last visit of the  $j^{\text{th}}$  walker to  $o$  before  $s_i$ . By definition, this means that  $r < s_i - 2^i$ . We can now write

$$\begin{aligned} \mathbb{P}(B_i(j) \mid s_i, r) &= \mathbb{P}(\text{a walker returned at time } s_i - r + n \\ &\quad \mid \text{not returning in the first } s_i - r \text{ steps}) \\ &\leq \sum_{j=0}^n \mathbb{P}(\text{a walker returned at time } s_i - r + j \\ &\quad \mid \text{not returning in the first } s_i - r + j - 1 \text{ steps}) \\ \text{by [3]} \quad &\leq \sum_{j=1}^n C \frac{\log(s_i - r + j)}{s_i - r + j} \leq \frac{Cni}{2^i} \end{aligned}$$

Integrating over  $r$  and  $s_i$  gives that  $\mathbb{P}(B_i(j)) \leq Cni2^{-i}$ , and summing over  $j$  (which has  $k$  possibilities, where  $k$  is the (unknown) number of walkers) gives  $\mathbb{P}(B_i) \leq Ckni2^{-i}$ . We see that these numbers are summable, so only a finite number of  $B_i$  occur. This means that  $p(n)$  may be calculated by

$$p(n) = \lim_{\ell \rightarrow \infty} \frac{|\{i \leq \ell : E_i\}|}{\ell}$$

which is the algorithm sought for.  $\square$

With the distribution of returns estimated, we now have a relatively easy task: we have a known variable, the number of returns of a single walker. We are given a sample of a union of  $k$  independent copies of it and we need to estimate  $k$ . Taking the number of actual returns up to time  $t$  and dividing by the (known) expectation for a single walker, would give a variable with expectation  $k$ . It would be natural to assume that if we repeat this experiment with times  $t_i$  growing sufficiently fast, the resulting variables would be approximately independent and hence it would be possible to calculate  $k$  by the limit of the running average. The only

difficulty is to explain what does “sufficiently fast” means, and it turns out that this must depend on the graph. The following lemma essentially claims that this scheme works if one takes  $t_i$  to be the median of the  $(2^i)^{\text{th}}$  return of a single walker to  $o$ .

**Lemma 2.** *Let  $X_n$  be i.i.d.  $\mathbb{N}$ -valued random variables, and let  $S_n$  be the corresponding random walk*

$$S_n = X_1 + \dots + X_n.$$

*Let  $M_n$  be the median of  $S_n$  i.e.*

$$M_n = \text{Med}(S_n) \implies n = \max\{i : \mathbb{P}(S_i \leq M_n) > \tfrac{1}{2}\}.$$

*Define*

$$Y_n = \frac{1}{n} \max\{i : S_i \leq M_n\}$$

*Then*

$$\frac{1}{N} \sum_{n=1}^N \frac{Y_{2^n}}{\mathbb{E}Y_{2^n}} \rightarrow 1$$

*as  $N \rightarrow \infty$ .*

*Proof.* By definition,  $\text{Med } Y_n = 1$ . It is easy to conclude from that that  $Y_n$  form a precompact (tight) family of variables. Indeed,

$$Y_n > \lambda \iff \max\{i : S_i \leq M_n\} > n\lambda \iff S_{\lfloor n\lambda \rfloor + 1} \leq M_n.$$

Since  $S$  is an increasing random walk, to be smaller than  $M_n$  at time  $n\lambda$  its increments must be smaller than  $M_n$  on any block of variables between 0 and  $n\lambda$ , and disjoint blocks are independent. So we get

$$\mathbb{P}(Y_n > \lambda) = \mathbb{P}(S_{\lfloor n\lambda \rfloor + 1} \leq M_n) \leq \mathbb{P}(X_{(k-1)n+1} + \dots + X_{kn} \leq M_n \forall k \leq \lfloor \lambda \rfloor) \leq 2^{-\lfloor \lambda \rfloor}. \quad (1)$$

We will use the second moment method so we need to estimate

$$\mathbb{E}(Y_n Y_m) - \mathbb{E}(Y_n) \mathbb{E}(Y_m)$$

say, for  $m < n$  (both will be powers of two but let us not record this fact in the notation). Let therefore  $\lambda = \lfloor (n/m)^{1/3} \rfloor$ . Define the event  $\mathcal{B}$  (the “bad” event) to be the event that one of the following happened:

1.  $Y_m \geq \lambda$ .
2.  $S_{mY_m+1} > M_{m\lambda^2}$  (note the  $+1$  in the index — we are taking here the first time  $S_i$  raises above  $M_m$ ).

The probability of the first clause is estimated by (1) to be  $\leq 2^{-\lambda}$ , so let us estimate the probability of the second minus the first, i.e. the probability that  $Y_m < \lambda$  but  $S_{mY_m+1} > M_{m\lambda^2}$ . Because  $S$  is increasing, if  $mY_m < m\lambda$  then  $mY_m + 1 \leq m\lambda$  and then  $S_{mY_m+1} \leq S_{m\lambda}$  so we can write

$$\mathbb{P}(Y_m < \lambda, S_{mY_m+1} > M_{m\lambda^2}) \leq \mathbb{P}(S_{m\lambda} > M_{m\lambda^2}) =: p.$$

But then we can write

$$\begin{aligned} \frac{1}{2} &< \mathbb{P}(S_{m\lambda^2} \leq M_{m\lambda^2}) \\ &\leq \mathbb{P}(X_{(k-1)m\lambda+1} + \dots + X_{km\lambda} \leq M_{m\lambda^2} \quad \forall k \leq \lambda) \\ &\leq (1-p)^\lambda \end{aligned}$$

so  $p \leq C/\lambda$ . Totally we get

$$\mathbb{P}(\mathcal{B}) \leq C/\lambda.$$

With the estimate (1) this gives

$$\mathbb{E}(Y_m \mathbf{1}_{\mathcal{B}}) \leq \sum_{k=1}^{\infty} \mathbb{P}(\mathcal{B} \cap \{k-1 < Y_m \leq k\}) \cdot k \leq \sum_{k=1}^{\infty} Ck \min\left\{\frac{1}{\lambda}, 2^{-k}\right\} \leq \frac{C(\log \lambda)^2}{\lambda}. \quad (2)$$

We need a similar estimate for  $Y_n Y_m \mathbf{1}_{\mathcal{B}}$  and for this we need to estimate  $\mathbb{E}(Y_n | \mathcal{B} \cap \{k-1 < Y_m \leq k\})$ . We write  $nY_n = mY_m + 1 + Z$  and note that  $Z$  is the number of steps our random walk needed to get from  $S_{mY_m+1}$  to  $n$  so it is stochastically dominated by  $nY_n$ , even after conditioning over  $\mathcal{B} \cap \{k-1 < Y_m \leq k\}$  (which is an event that looks at the random walk only up to  $mY_m + 1$ ). So

$$\mathbb{E}(Y_n | \mathcal{B} \cap \{Y_m = y\}) \leq y \frac{m}{n} + C$$

which we use to show

$$\begin{aligned} \mathbb{E}(Y_n Y_m \mathbf{1}_{\mathcal{B}}) &\leq \sum_{k=1}^{\infty} \mathbb{P}(\mathcal{B} \cap \{k-1 < Y_m \leq k\}) \cdot k \cdot (k(m/n) + C) \\ &\leq \sum_{k=1}^{\infty} Ck^2 \min\left\{\frac{1}{\lambda}, 2^{-k}\right\} \leq \frac{C(\log \lambda)^3}{\lambda}. \end{aligned} \quad (3)$$

This finishes our treatment of the event  $\mathcal{B}$ .

We now restrict our attention to  $\neg \mathcal{B}$ . Let therefore  $\omega$  be some atom of the  $\sigma$ -field spanned by  $Y_m, X_1, \dots, X_{mY_m+1}$  such that  $\omega \notin \mathcal{B}$ , and write

$$\mathbb{E}(Y_n | \omega) = Y_m \cdot \frac{m}{n} + \frac{1}{n} \mathbb{E}(\max\{i : X_{mY_m+2} + \dots + X_{mY_m+1+i} \leq M_n - (S_{mY_m+1})\} | \omega) \quad (4)$$

The first term we bound by  $C/\lambda^2$  (because of the first clause in the definition of  $\mathcal{B}$ ). For the second, we note that  $X_{2mY_m+2}, \dots$  has the same distribution as  $X_1, \dots$  (again, conditioning

over  $\omega$  does not change this fact) so this term is bounded above by  $\mathbb{E}Y_n$  and bounded below by

$$\frac{1}{n}\mathbb{E}(\max\{i : S_i \leq M_n - M_{m\lambda^2}\})$$

by the second clause in the definition of  $\mathcal{B}$ . Hence we need to estimate the variable

$$\max\{i : S_i \leq M_n\} - \max\{i : S_i \leq M_n - M_{m\lambda^2}\} = |\{i : M_n - M_{m\lambda^2} < S_i \leq M_n\}|$$

But this variable is stochastically dominated simply by  $m\lambda^2 Y_{m\lambda^2}$  because it is the number of steps our random walk needs to traverse an interval  $\leq M_{m\lambda^2}$ . Combining both parts of (4) gives

$$\mathbb{E}(Y_n) - C/\lambda \leq \mathbb{E}(Y_n|\omega) \leq C/\lambda + \mathbb{E}(Y_n)$$

which we multiply by  $Y_m$  and integrate over  $\neg\mathcal{B}$  to get

$$|\mathbb{E}(Y_m Y_n \mathbf{1}_{\neg\mathcal{B}}) - \mathbb{E}(Y_n)\mathbb{E}(Y_m \mathbf{1}_{\neg\mathcal{B}})| \leq \frac{C}{\lambda}\mathbb{E}(Y_m \mathbf{1}_{\neg\mathcal{B}}) \leq \frac{C}{\lambda}.$$

With (2), (3) we get

$$|\mathbb{E}(Y_m Y_n) - \mathbb{E}(Y_m)\mathbb{E}(Y_n)| \leq \frac{C(\log \lambda)^3}{\lambda}. \quad (5)$$

This finishes the lemma: define

$$A_N = \sum_{i=1}^N Y_{2^i}$$

and estimate  $\mathbb{V}A_N$ . We get

$$\begin{aligned} \mathbb{V}A_N &= \sum_{i=1}^N \mathbb{V}Y_{2^i} + 2 \sum_{1 \leq i < j \leq N} \text{cov}(Y_{2^i}, Y_{2^j}) \\ &\leq \sum_{i=1}^N C + 2 \sum_{1 \leq i < j \leq N} C \cdot 2^{(i-j)/3} |i - j|^3 \leq CN \end{aligned}$$

where the bound for  $\mathbb{V}Y_{2^i}$  comes from the exponential decay (1), and the bound for the covariances is exactly (5), recall that  $\lambda$  was defined by  $\lfloor (n/m)^{1/3} \rfloor$ . On the other hand,  $\text{Med } Y_{2^i} = 1$  so  $\mathbb{E}Y_{2^i} \geq \frac{1}{2}$  and  $\mathbb{E}A_N \geq \frac{1}{2}N$ . This gives that  $A_N/\mathbb{E}A_N$  is concentrated. Using Markov's inequality gives

$$\mathbb{P}\left(\left|\frac{A_N}{\mathbb{E}A_N} - 1\right| > \epsilon\right) \leq \mathbb{P}(|A_N - \mathbb{E}A_N| > c\epsilon N) \leq \frac{C}{\epsilon^2 N}.$$

This means that these events happen only finitely many times on any reasonable subsequence (e.g.  $N^2$ ) and due to  $\mathbb{E}Y_{2^i} \approx 1$  and monotonicity of  $A_N$  the convergence may be extended from a subsequence to all  $N$ .  $\square$

We are almost done, we just need to handle double returns to  $o$ , for which we have the following simple lemma.

**Lemma 3.** *Let  $X_1$  and  $X_2$  be two independent walkers on an infinite graph  $G$ , and let  $t > 0$ . Then*

$$\mathbb{E}(|\{s \leq t : X_1(s) = X_2(s) = o\}|) \leq C \left( \mathbb{E}(|\{s \leq t : X_1(s) = o\}|) \right)^{2/3}.$$

*Proof.* Denote  $M = \mathbb{E}(|\{s \leq t : X_1(s) = o\}|)$ . On any infinite graph,  $\mathbb{P}(X_2(s) = o) \leq C/\sqrt{s}$  (see e.g. [2]). Hence we write

$$\begin{aligned} \mathbb{E}(|\{M^{2/3} \leq s \leq t : X_1(s) = X_2(s) = o\}|) &= \sum_{s=M^{2/3}}^t \mathbb{P}(X_1(s) = o)^2 \\ &\leq \sum_{s=M^{2/3}}^t \mathbb{P}(X_1(s) = o) \cdot \frac{C}{\sqrt{s}} \leq CM^{-1/3} \sum_{s=1}^t \mathbb{P}(X_1(s) = o) = CM^{2/3}. \end{aligned}$$

Since the number of visits up to time  $M^{2/3}$  is definitely bounded by  $M^{2/3}$ , we are done.  $\square$

The theorem now follows easily. By lemma 1 we may calculate the median  $M_n$  of the  $n^{\text{th}}$  return of a single walker to  $o$ . Defining

$$Y_n^i = \frac{1}{n} |\{\text{visits of walker } i \text{ until } M_n\}| \quad Y_n = \sum_i Y_n^i$$

we can use lemma 2 (the random walk  $S$  in lemma 2 is defined by  $S_n$  being the time of the  $n^{\text{th}}$  visit of the walker to  $o$  and then the  $M_n$  of lemma 2 are the same as here, and the  $Y_n$  of lemma 2 are the  $Y_n^i$  here). We get

$$\frac{1}{N} \sum_{n=1}^N \frac{Y_{2^n}}{\mathbb{E}Y_{2^n}^i} \rightarrow k.$$

We cannot measure  $Y_n$  directly, since if two walkers returned to  $o$  at the same time, they contribute 2 to the sum but we cannot see that. Nevertheless, if we define

$$\tilde{Y}_n = \frac{1}{n} |\{1 \leq t \leq M_n : \exists j, X_j(t) = o\}|$$

then  $\tilde{Y}_n$  can be measured, and

$$|Y_n - \tilde{Y}_n| \leq \frac{1}{n} \sum_{i,j} |\{1 \leq t \leq M_n : X_i(t) = X_j(t) = o\}|$$

and each term is bounded by lemma 3 by  $(\mathbb{E}(nY_n))^{2/3}/n$ . Since  $\mathbb{E}Y_n \leq C$  this gives that

$$|Y_n - \tilde{Y}_n| \leq Ck^2 n^{-1/3}.$$

We get

$$\frac{1}{N} \sum_{n=1}^N \frac{\tilde{Y}_{2^n}}{\mathbb{E}Y_{2^n}^i} \rightarrow k$$

and the theorem is proved.  $\square$

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## References

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